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# Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings

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**Abstract**

In this paper, it is our purpose to introduce an iterative process for the approximation of a common fixed point of a finite family of multi-valued Bregman relatively nonexpansive mappings. We prove that the sequence of iterates generated converges strongly to a common fixed point of a finite family of multi-valued Bregman nonexpansive mappings in reflexive real Banach spaces.

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**Keywords:** Bregman projection; Legendre function; multi-valued Bregman nonexpansive mapping; relatively nonexpansive multi-valued mapping; single-valued Bregman nonexpansive mapping; strong convergence

## 1 Introduction

Let $E$ be a reflexive real Banach space $E$, and $E^*$ its dual. Let $f : E \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. The *subdifferential of $f$ at $x \in E$* is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in E\}. \tag{1.1}$$

The *Fenchel conjugate* of $f$ is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(y) = \sup\{\langle y, x\rangle - f(x) : x \in E\}$. It is not difficult to check that when $f$ is proper and lower semicontinuous, so is $f^*$.

The function $f$ is said to be *essentially smooth* if $\partial f$ is both locally bounded and single-valued on its domain. It is called *essentially strictly convex*, if $(\partial f)^{-1}$ is locally bounded on its domain and $f$ is strictly convex on every convex subset of $\operatorname{dom} \partial f$. $f$ is said to be *Legendre*, if it is both essentially smooth and essentially strictly convex.

Let $\operatorname{dom} f = \{x \in E : f(x) < \infty\}$. Then for any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, the *right-hand derivative of $f$ at $x$* in the direction of $y$ is defined by

$$f^\circ(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{1.2}$$

If the limit in (1.2) exists then $f$ is called *Gâteaux differentiable* at $x$. In this case, $f^\circ(x, y)$ coincides with $\nabla f(x)$, the value of the gradient $\nabla f$ of $f$ at $x$. The function $f$ is called *Gâteaux* *differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. The function $f$ called *Fréchet differentiable* at $x$ if the limit in (1.2) is attained uniformly for all $y \in E$ such that $\|y\| = 1$ and $f$ is said to be *uniformly Fréchet differentiable* on a subset $C$ of $E$ if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. When the subdifferential of $f$ is single-valued, it coincides with the gradient $\partial f = \nabla f$ (see [1]).

We remark that if $E$ is a reflexive Banach space. Then we have

(1) $f$ is essentially smooth if and only if $f^*$ is essentially strictly convex (see [2], Theorem 5.4).
(2) $(\partial f)^{-1} = \partial f^*$ (see [3]).
(3) $f$ is Legendre if and only if $f^*$ is Legendre (see [2], Corollary 5.5).
(4) If $f$ is Legendre, then $\nabla f$ is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}(\operatorname{dom} f)$ (see [2], Theorem 5.10).

When $E$ is a smooth and strictly convex Banach space, one important and interesting example of Legendre function is $f(x) := \|x\|^p$ $(1 < p < \infty)$. In this case the gradient $\nabla f = pJ_p$ $(1 < p < \infty)$, where $J_p$ is the generalized duality mapping from $E$ into $2^{E^*}$ defined by

$$J_p(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^p, \|f^*\| = \|x\|^{p-1}\}.$$

In particular, $J = J_2$ is called the *normalized duality mapping*. It is well known that if $E^*$ is strictly convex, then $J_p$ is single-valued and that

$$J_p(x) = \|x\|^{p-2} J(x), \quad x \neq 0.$$

If $E = H$, a Hilbert space, then $J$ is the identity mapping and hence $\nabla f = 2I$, where $I$ is the identity mapping in $H$.

In this paper, $E$ is a reflexive real Banach space, $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function, and $f^* : E^* \to (-\infty, +\infty]$ is the Fenchel conjugate of $f$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, +\infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle$$

is called the *Bregman distance with respect to* $f$ [4]. Since $(\nabla f)^{-1} = \nabla f^*$ and $f^*(\nabla f) = \langle x, \nabla f(x) \rangle - f(x)$, it is easy to check that

$$D_{f^*}(\nabla f(y), \nabla f(x)) = D_f(x, y). \tag{1.3}$$

A *Bregman projection* [5] of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty closed and convex set $C \subset \operatorname{int}(\operatorname{dom} f)$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

**Remark 1.1** If $E$ is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have $\nabla f(x) = 2Jx$, for all $x \in E$, where $J$ the normalized duality mapping and hence the function $D_f(x,y)$ reduces to $\phi(x,y)$ which is defined by $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$, which is the Lyapunov function introduced by Alber [6], and $P^f_C(x)$ reduces to the generalized projection $\Pi_C(x)$ (see, *e.g.*, [6]), which is defined by

$$\phi\bigl(\Pi_C(x), x\bigr) = \min_{y \in C} \phi(y,x).$$

If $E = H$, a Hilbert space, then $J$ is the identity mapping and hence the Bregman distance becomes $D_f(x,y) = \|x-y\|^2$, for $x, y \in H$, and the Bregman projection $P^f_C(x)$ reduces to the metric projection $P_C$ of $H$ on to $C$.

Let $C$ be a nonempty closed and convex subset of $\operatorname{int}(\operatorname{dom}(f))$. Let $T : C \to \operatorname{int}(\operatorname{dom}(f))$ be a mapping. An element $p \in C$ is called a *fixed point* of $T$ if $T(p) = p$. The set of fixed points of $T$ is denoted by $F(T)$. A point $p$ in $C$ is said to be an *asymptotic fixed point of* $T$ (see [7]) if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\widehat{F}(T)$. $T$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$, and is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$. The mapping $T$ is called *relatively nonexpansive* if (A1) $F(T) \neq \emptyset$; (A2) $\phi(p, Tx) \leq \phi(p,x)$ for $x \in C$ and $p \in F(T)$, and (A3) $F(T) = \widehat{F}(T)$ and is said to be *Bregman relatively nonexpansive* with respect to $f$ if (B1) $F(T) \neq \emptyset$; (B2) $D_f(p, Tx) \leq D_f(p,x)$ for $x \in C$, $p \in F(T)$ and (B3) $F(T) = \widehat{F}(T)$. We remark that the class of relatively nonexpansive mappings is contained in a class of Bregman relatively nonexpansive mappings with respect to $f(x) = \|x\|^2$.

Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of $C$, respectively. Let $H$ be the Hausdorff metric on $CB(C)$ defined by

$$H(A,B) = \max\Bigl\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\Bigr\},$$

for all $A, B \in CB(C)$, where $d(a,B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point $a$ to the subset $B$.

Let $T : C \to CB(C)$ be a mapping. $T$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in C$. An element $p \in C$ is called a *fixed point* of $T$, if $p \in F(T)$, where $F(T) := \{p \in C : p \in T(p)\}$. A point $p \in C$ is called an *asymptotic fixed point* of $T$, if there exists a sequence $\{x_n\}$ in $C$ which converges weakly to $p$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. $T$ is called *relatively nonexpansive* if (A1)′ $F(T) \neq \emptyset$; (A2)′ $\phi(p,u) \leq \phi(p,x)$ for all $u \in Tx$, $x \in C$, and (A3)′ $F(T) = \widehat{F}(T)$. A mapping $T$ is called *quasi-Bregman nonexpansive* with respect to $f$ if $F(T) \neq \emptyset$ and $D_f(p,u) \leq D_f(p,x)$ for all $u \in Tx$, $x \in C$, $p \in F(T)$ and is called *Bregman relatively nonexpansive* with respect to $f$ if (B1)′ $F(T) \neq \emptyset$; (B2)′ $D_f(p,u) \leq D_f(p,x)$ for $u \in Tx$, $x \in C$, $p \in F(T)$, and (B3)′ $F(T) = \widehat{F}(T)$.

We note that the class of multi-valued relatively nonexpansive mappings is contained in a class of multi-valued Bregman relatively nonexpansive mappings which includes the class of single-valued Bregman relatively nonexpansive mappings. Hence, the class of multi-valued Bregman relatively nonexpansive mappings is a more general class of mappings. An example of a multi-valued Bregman relatively nonexpansive mapping is given now.

**Example 1.2** Let $I = [0,1]$, $X = L^p(I)$, $1 < p < \infty$ and $C = \{f \in X : f(x) \geq 0, \forall x \in I\}$. Let $T : C \to CB(C)$ be defined by

$$T(f) = \begin{cases} \{h \in C : f(x) - \frac{1}{2} \leq h(x) \leq f(x) - \frac{1}{4}, \forall x \in I\}, & \text{if } f(x) > 1, \forall x \in I; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let $g : E \to \mathbb{R}$ be defined by $g(x) = \frac{1}{p}\|x\|^p$, $1 < p < \infty$, $x \in E$. Clearly, we have $\nabla g(x) = J_p(x)$ for all $x \in E$, and $g^*(x^*) = \frac{1}{q}\|x^*\|^q$, where $1 < q < \infty$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. It is clear that $F(T) = \{0\}$. Let $f \in C$ and $h \in T(f)$ such that $f(x) > 1$ for all $x \in [0,1]$. Then, using (1.3), we get

$$\begin{aligned} D_g(0,h) &= D_{g^*}\big(\nabla g(h), \nabla g(0)\big) = D_{g^*}\big(J_p(h), 0\big) \\ &= g^*\big(J_p(h)\big) - g^*(0) - \big\langle \nabla g^*(0), J_p(h) - 0\big\rangle \\ &= \frac{1}{q}\big\|J_p(h)\big\|^q \leq \frac{1}{q}\big\|J_p(f)\big\|^q \\ &= D_{g^*}\big(J_p(f), 0\big) = D_g(0,f). \end{aligned}$$

Next, let $f \in C$ such that there exists $x \in I$ such that $f(x) \leq 1$, then

$$D_g(0,0) = D_{g^*}(0,0) \leq D_{g^*}\big(J_p(f), 0\big) = D_g(0,f).$$

Hence, $T$ is a multi-valued quasi-Bregman nonexpansive mapping. Now, we show that $\widehat{F}(T) = F(T)$. Let $\{f_n\} \subset C$ be a sequence which converges weakly to $h$, and $z_n = d(f_n, T(f_n)) \to 0$. Let $n \in \mathbb{N}$, then we have

$$z_n = \begin{cases} \frac{1}{4}, & \text{if } f_n(x) > 1, \forall x \in [0,1]; \\ \|f_n\|_p, & \text{otherwise.} \end{cases}$$

Since $z_n \to 0$, we have $\|f_n\|_p \to 0$ and hence $h = 0$. Therefore, $T$ is a multi-valued Bregman relatively nonexpansive mapping.

The approximations of fixed points of nonexpansive, quasi-nonexpansive, relatively nonexpansive, and relatively quasi-nonexpansive mappings when they exist have been intensively studied for almost 40 years or so by various authors (see, *e.g.*, [8–18] and the references therein) in Banach spaces.

In 1967, Bregman [5] discovered an effective technique using the so-called Bregman distance function $D_f$ in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms for solving not only feasibility and optimization problems, but also algorithms for solving variational inequality problems, equilibrium problems, fixed point problems for nonlinear mappings, and so on (see, *e.g.*, [7, 19, 20], and the references therein).

In [21], Reich and Sabach proposed the following algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators defined on a nonempty, closed and convex subset $C$ of a reflexive Banach space $E$ (see also [22, 23]). The construction of fixed points for Bregman-type single-valued mappings via iterative processes has been investigated in, for example, [21, 24–27]. This now leads to the following important question.

**Question** Is it possible to obtain the results of Reich and Sabach [21] for the class of multi-valued Bregman relatively nonexpansive mappings?

The study of fixed points for multi-valued nonexpansive mappings using the Hausdorff metric was introduced by Markin [28] (see also [29]). Later, an interesting and rich fixed point theory for such mappings was developed which has applications in control theory, convex optimization, differential inclusion, and economics (see, for example, [30] and references therein). Moreover, the existence of fixed points for multi-valued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [31] (see also [32]).

Recently, Homaeipour and Razani [33] studied the following iterative scheme for a fixed point of relatively nonexpansive multi-valued mapping in uniformly convex and uniformly smooth Banach space $E$:

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrary,} \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \quad z_n \in Tx_n, n \geq 0, \end{cases} \tag{1.4}$$

where $\{\alpha_n\} \subset (0,1)$ for all $n \geq 0$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. They proved that if $J$ is weakly sequentially continuous then the sequence $\{x_n\}$ converges *weakly* to a fixed point of $T$. Furthermore, it is shown that the scheme converges strongly to a fixed point of $T$ if the *interior of* $F(T)$ *is nonempty*.

More recently, Zegeye and Shahzad [34], extended the above result to a finite family of multi-valued relatively nonexpansive mappings without the assumption that the *interior of* $F(T)$ *is nonempty*. In fact, they proved that if $C$ is a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex real Banach space $E$ and $T_i : C \to CB(C)$, for $i = 1, 2, \ldots, N$, are relatively nonexpansive multi-valued mappings with $\mathcal{F} := \bigcap_{i=1}^N F(T_i)$ nonempty, then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = w \in C, \quad \text{chosen arbitrarily,} \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jx_n), \\ x_{n+1} = J^{-1}(\beta_{n,0}Jx_n + \sum_{i=1}^N \beta_{n,i}Ju_{n,i}), \quad u_{n,i} \in T_i y_n, n \geq 0, \end{cases}$$

where $\alpha_n \in (0,1)$ and $\{\beta_{n,i}\} \subset [a,b] \subset (0,1)$, for $i = 1, 2, \ldots, N$, satisfy certain conditions, converges strongly to an element of $\mathcal{F}$.

In this paper, it is our purpose to introduce an iterative scheme which converges strongly to a common fixed point of a finite family of multi-valued Bregman relatively nonexpansive mappings. We prove strong convergence theorems for the sequences produced by the method. Our results improve and generalize many known results in the current literature (see, for example, [33, 34]).

## 2 Preliminaries

Let $E$ be a reflexive real Banach space and $E^*$ as its dual. Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of the total convexity of $f$ at $x \in \operatorname{dom} f$ is the function $\nu_f(x,\cdot) : [0,+\infty) \to [0,+\infty]$ defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in \operatorname{dom} f, \|y-x\| = t\}.$$

The function $f$ is called *totally convex* at $x$ if $\nu_f(x,t) > 0$, whenever $t > 0$. The function $f$ is called *totally convex* if it is totally convex at any point $x \in \operatorname{int}(\operatorname{dom} f)$ and is said to be *totally convex on bounded sets* if $\nu_f(B,t) > 0$ for any nonempty bounded subset $B$ of $E$ and $t > 0$, where the modulus of total convexity of the function $f$ on the set $B$ is the function $\nu_f : \operatorname{int}\operatorname{dom} f \times [0,+\infty) \to [0,+\infty]$ defined by

$$\nu_f(B,t) := \inf\{V_f(x,t) : x \in B \cap \operatorname{dom} f\}.$$

Let $E$ be a Banach space and let $B_r := \{z \in E : \|z\| \le r\}$ for all $r > 0$ and $S_E = \{x \in E : \|x\| = 1\}$. Then a function $f : E \to \mathbb{R}$ is said to be *uniformly convex* on bounded subsets of $E$ [35, pp.203] if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0,\infty) \to [0,\infty]$ is defined by

$$\rho_r(t) := \inf_{x,y\in B_r, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \ge 0$. The function $\rho_r$ is called the *gauge of the uniform convexity of $f$*. We know that $f$ is totally convex on bounded sets if and only if $f$ is uniformly convex on bounded sets (see [36], Theorem 2.10).

If $f$ is uniformly convex, then the following lemma is known.

**Lemma 2.1** [37] *Let $E$ be a Banach space, let $r > 0$ be a constant, and let $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of $E$. Then*

$$f\left(\sum_{k=0}^{n} \alpha_k x_k\right) \le \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - y_j\|),$$

*for all $i, j \in \{0,1,2,\ldots,n\}$, $x_k \in Br$, $\alpha_k \in (0,1)$, and $k = 0,1,2,\ldots,n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where $\rho_r$ is the gauge of uniform convexity of $f$.*

A function $f$ on $E$ is *coercive* [38] if the sublevel set of $f$ is bounded; equivalently, $\lim_{\|x\|\to\infty} f(x) = \infty$. A function $f$ on $E$ is said to be *strongly coercive* [35] if $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$.

In the sequel, we shall need the following lemmas.

**Lemma 2.2** [39] *The function $f : E \to (-\infty,+\infty)$ is totally convex on bounded subsets of $E$ if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in* $\operatorname{int}(\operatorname{dom} f)$ *and* $\operatorname{dom} f$, *respectively, such that the first one is bounded, we have*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

**Lemma 2.3** [35] *Let $f : E \to \mathbb{R}$ be a strongly coercive and uniformly convex on bounded subsets of $E$, then $f^*$ is bounded and uniformly Fréchet differentiable on bounded subsets of $E^*$.*

**Lemma 2.4** [26] *Let $f : E \to (-\infty,+\infty]$ be a uniformly Fréchet differentiable and bounded on bounded sets of $E$, then $\nabla f$ is uniformly continuous on bounded subsets of $E$ from the strong topology of $E$ to the strong topology of $E^*$.*

**Lemma 2.5** [1] *Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak$^*$ lower semicontinuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i). \tag{2.1}$$

**Lemma 2.6** [40] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable on* $\operatorname{int}(\operatorname{dom} f)$ *such that $\nabla f^*$ is bounded on bounded subsets of* $\operatorname{dom} f^*$. *Let $x \in E$ and $\{x_n\} \subset E$. If $\{D_f(x, x_n)\}$ is bounded, so is the sequence $\{x_n\}$.*

**Lemma 2.7** [36] *Let $C$ be a nonempty, closed, and convex subset of $E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(i) $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0$, $\forall y \in C$.
(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x)$, $\forall y \in C$.

Let $f : E \to \mathbb{R}$ be a Legendre and Gâteaux differentiable function. Following [6] and [4], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with $f$, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*. \tag{2.2}$$

Then we observe that $V_f$ is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \quad \text{for all } x \in E \text{ and } x^* \in E^*. \tag{2.3}$$

Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \tag{2.4}$$

$\forall x \in E$ and $x^*, y^* \in E^*$ (see [41]).

**Lemma 2.8** [42] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, \quad n \geq n_0,$$

*where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the following conditions*: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.9** [43] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all* (*sufficiently large*) *numbers $k \in \mathbb{N}$*:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

*In fact, $m_k$ is the largest number $n$ in the set $\{1, 2, \ldots, k\}$ such that the condition $a_n \leq a_{n+1}$ holds.*

## 3 Main result

In the sequel we shall use the following proposition.

**Proposition 3.1** *Let $f : E \to \mathbb{R}$ be a uniformly Fréchet differentiable and totally convex on bounded subsets of $E$. Let $C$ be a nonempty, closed, and convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $T : C \to CB(C)$ be a Bregman relatively nonexpansive mapping. Then $F(T)$ is closed and convex.*

*Proof* First, we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x^*$. Since $T$ is Bregman relatively nonexpansive mapping, we have $D_f(x_n, u) \leq D_f(x_n, x^*)$, for all $u \in Tx^*$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} D_f\left(x^*, u\right) &= \lim_{n\to\infty} D_f(x_n, u) \\ &\leq \lim_{n\to\infty} D_f\left(x_n, x^*\right) \\ &= D_f\left(x^*, x^*\right) = 0. \end{aligned} \tag{3.1}$$

Thus, by Lemma 2.2 we obtain $x^* = u$. Hence, $x^* \in F(T)$ and $F(T)$ is closed. Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$ and $p = tx + (1-t)y$ for $t \in (0,1)$. We show that $p \in F(T)$. Let $w \in T(p)$, then we have

$$\begin{aligned} D_f(p, w) &= f(p) - f(w) - \langle \nabla f(w), p - w \rangle \\ &= f(p) - f(w) - \langle \nabla f(w), tx + (1-t)y - w \rangle \\ &= f(p) + tD_f(x, w) + (1-t)D_f(y, w) - tf(x) - (1-t)f(y) \\ &\leq f(p) + tD_f(x, p) + (1-t)D_f(y, p) - tf(x) - (1-t)f(y) \\ &\leq f(p) + t\bigl[f(x) - f(p) - \langle \nabla f(p), x - p \rangle\bigr] \\ &\quad + (1-t)\bigl[f(y) - f(p) - \langle \nabla f(p), y - p \rangle\bigr] - tf(x) - (1-t)f(y) \\ &= \langle \nabla f(p), tx + (1-t)y - p \rangle = 0. \end{aligned}$$

Thus, by Lemma 2.2 we get $p \in T(p)$. Hence, $p \in F(T)$ and $F(T)$ is convex. Therefore, $F(T)$ is closed and convex. □

**Theorem 3.2** *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $E$. Let $C$ be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $T_i : C \to CB(C)$, for $i = 1, 2, \ldots, N$, be a finite family of Bregman relatively nonexpansive mappings such that $\mathcal{F} := \bigcap_{i=1}^N F(T_i)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} w_n = P_C^f \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)), \\ x_{n+1} = \nabla f^*(\beta_0 \nabla f(w_n) + \sum_{i=1}^N \beta_i \nabla f(u_{i,n})), \quad u_{i,n} \in T_i w_n, \forall n \geq 0, \end{cases} \tag{3.2}$$

*where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_i\}_{i=0}^N \subset [c,d] \subset (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{i=0}^N \beta_i = 1$. Then $\{x_n\}$ converges strongly to $p = P_{\mathcal{F}}^f(u)$.*

*Proof* Proposition 3.1 ensures that each $F(T_i)$, for $i \in \{1, 2, \ldots, N\}$, and hence $\mathcal{F}$, is closed and convex. Thus, $P^f_{\mathcal{F}}$ is well defined. Let $p = P^f_{\mathcal{F}}(u)$. Then, from (3.2), Lemmas 2.7, 2.5, and the property of $D_f$, we get

$$\begin{aligned} D_f(p, w_n) &= D_f\big(p, P^f_C \nabla f^*\big(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)\big)\big) \\ &\le D_f\big(p, \nabla f^*\big(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)\big)\big) \\ &= \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n). \end{aligned} \tag{3.3}$$

Moreover, from (3.2), (2.3), and (2.2) we get

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f\left(p, \nabla f^*\left(\beta_0 \nabla f(w_n) + \sum_{i=1}^N \beta_i \nabla f(u_{i,n})\right)\right) \\ &= V_f\left(p, \beta_0 \nabla f(w_n) + \sum_{i=1}^N \beta_i \nabla f(u_{i,n})\right) \\ &= f(p) - \left\langle p, \beta_0 \nabla f(w_n) + \sum_{i=1}^N \beta_i \nabla f(u_{i,n})\right\rangle \\ &\quad + f^*\left(\beta_0 \nabla f(w_n) + \sum_{i=1}^N \beta_i \nabla f(u_{i,n})\right). \end{aligned}$$

Since $f$ is uniformly Fréchet differentiable function we find that $f$ is uniformly smooth and hence by Theorem 3.5.5 of [35] we find that $f^*$ is uniformly convex. This, with Lemma 2.1 and (3.3), gives

$$\begin{aligned} D_f(p, x_{n+1}) &\le f(p) - \beta_0\langle p, \nabla f(w_n)\rangle - \sum_{i=1}^N \beta_i \langle p, \nabla f(u_{i,n})\rangle \\ &\quad + \beta_0 f^*\big(\nabla f(w_n)\big) + \sum_{i=1}^N \beta_i f^*\big(\nabla f(u_{i,n})\big) \\ &\quad - \beta_0\beta_i\rho_r^*\big(\big\|\nabla f(w_n) - \nabla f(u_{i,n})\big\|\big) \\ &= \beta_0 V\big(p, \nabla f(w_n)\big) + \sum_{i=1}^N \beta_i V\big(p, \nabla f(u_{i,n})\big) \\ &\quad - \beta_0\beta_i\rho_r^*\big(\big\|\nabla f(w_n) - \nabla f(u_{i,n})\big\|\big) \\ &= \beta_0 D_f(p, w_n) + \sum_{i=1}^N \beta_i D_f(p, u_{i,n}) \\ &\quad - \beta_0\beta_i\rho_r^*\big(\big\|\nabla f(w_n) - \nabla f(u_{i,n})\big\|\big) \\ &\le \beta_0 D_f(p, w_n) + \sum_{i=1}^N \beta_i D_f(p, w_n) - \beta_0\beta_i\rho_r^*\big(\big\|\nabla f(w_n) - \nabla f(u_{i,n})\big\|\big) \\ &\le D_f(p, w_n) - \beta_0\beta_i\rho_r^*\big(\big\|\nabla f(w_n) - \nabla f(u_{i,n})\big\|\big) \le D_f(p, w_n) && (3.4) \\ &\le \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n), && (3.5) \end{aligned}$$

for each $i \in \{1, 2, \dots, N\}$. Thus, by induction,

$$D_f(p, x_{n+1}) \leq \max\{D_f(p, u), D_f(p, x_0)\}, \quad \forall n \geq 0,$$

which implies that $\{x_n\}$ is bounded. Furthermore, from (3.2), (2.3), (2.4), and Lemma 2.7 we obtain

$$\begin{aligned}
D_f(p, w_n) &= D_f\bigl(p, P_C^f \nabla f^*\bigl(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(x_n)\bigr)\bigr) \\
&\leq D_f\bigl(p, \nabla f^*\bigl(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(x_n)\bigr)\bigr) \\
&= V_f\bigl(p, \alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(x_n)\bigr) \\
&\leq V_f\bigl(p, \alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(x_n) - \alpha_n\bigl(\nabla f(u) - \nabla f(p)\bigr)\bigr) \\
&\quad + \alpha_n\bigl\langle \nabla f(u) - \nabla f(p), w_n - p\bigr\rangle \\
&= V_f\bigl(p, \alpha_n \nabla f(p) + (1 - \alpha_n)\nabla f(x_n)\bigr) \\
&\quad + \alpha_n\bigl\langle \nabla f(u) - \nabla f(p), w_n - p\bigr\rangle \\
&= D_f\bigl(p, \nabla f^*\bigl(\alpha_n \nabla f(p) + (1 - \alpha_n)\nabla f(x_n)\bigr)\bigr) \\
&\quad + \alpha_n\bigl\langle \nabla f(u) - \nabla f(p), w_n - p\bigr\rangle \\
&\leq \alpha_n D_f(p, p) + (1 - \alpha_n) D_f(p, x_n) \\
&\quad + \alpha_n\bigl\langle \nabla f(u) - \nabla f(p), w_n - p\bigr\rangle \\
&= (1 - \alpha_n) D_f(p, x_n) + \alpha_n\bigl\langle \nabla f(u) - \nabla f(p), w_n - p\bigr\rangle. \qquad (3.6)
\end{aligned}$$

Furthermore, from (3.4) and (3.6) we have

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq (1 - \alpha_n) D_f(p, x_n) + \alpha_n\bigl\langle \nabla f(u) - \nabla f(p), w_n - p\bigr\rangle \\
&\qquad - \beta_0 \beta_i \rho_r^*\bigl(\bigl\|\nabla f(w_n) - \nabla f(u_{i,n})\bigr\|\bigr) \qquad (3.7) \\
&\leq (1 - \alpha_n) D_f(p, x_n) + \alpha_n\bigl\langle \nabla f(u) - \nabla f(p), w_n - p\bigr\rangle. \qquad (3.8)
\end{aligned}$$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(p, x_n)\}$ is non-increasing for all $n \geq n_0$. In this situation, $\{D_f(p, x_n)\}$ is convergent. Then, from (3.7), we have

$$\beta_0 \beta_i \rho_r^*\bigl(\bigl\|\nabla f(w_n) - \nabla f(u_{i,n})\bigr\|\bigr) \to 0, \qquad (3.9)$$

which implies, by the property of $\rho_r^*$ that

$$\nabla f(w_n) - \nabla f(u_{i,n}) \to 0 \quad \text{as } n \to \infty. \qquad (3.10)$$

Now, since $f$ is strongly coercive and uniformly convex on bounded subsets of $E$ by Lemma 2.3 we see that $f^*$ is uniformly Fréchet differentiable on bounded subsets of $E^*$ and since $f$ is Legendre by Lemma 2.4 we find that $\nabla f^*$ is uniformly continuous on bounded subsets of $E^*$ and hence from (3.10) we get

$$w_n - u_{i,n} \to 0 \quad \text{as } n \to \infty. \qquad (3.11)$$

In addition, since $d(w_n, T_i w_n) \leq \|w_n - u_{i,n}\|$ we have

$$d(w_n, T_i w_n) \to 0 \quad \text{as } n \to \infty, \tag{3.12}$$

for each $i \in \{1, 2, \dots, N\}$. Since $\{w_n\}$ is bounded and $E$ is reflexive, we choose a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \rightharpoonup w$ and $\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), w_n - p\rangle = \lim_{j\to\infty} \langle \nabla f(u) - \nabla f(p), w_{n_j} - p\rangle$. Thus, from (3.12) and the fact that each $T_i$ is Bregman relatively nonexpansive mapping we obtain $w \in F(T_i)$, for each $i \in \{1, 2, \dots, N\}$ and hence $w \in \bigcap_{i=1}^N F(T_i)$.

Therefore, by Lemma 2.7, we immediately obtain

$$\begin{aligned}
\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), w_n - p\rangle &= \lim_{j\to\infty} \langle \nabla f(u) - \nabla f(p), w_{n_j} - p\rangle \\
&= \langle \nabla f(u) - \nabla f(p), w - p\rangle \leq 0.
\end{aligned}$$

It follows from Lemma 2.8 and (3.8) that $D_f(p, x_n) \to 0$ as $n \to \infty$. Consequently, by Lemma 2.2 we obtain $x_n \to p \in \mathcal{F}$.

Case 2. Suppose that there exists a subsequence $\{n_l\}$ of $\{n\}$ such that

$$D_f(p, x_{n_l}) < D_f(p, x_{n_l+1}),$$

for all $l \in \mathbb{N}$. Then, by Lemma 2.9, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1})$, and $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$, for all $k \in \mathbb{N}$. Then, from (3.7) and the fact that $\alpha_n \to 0$, we obtain

$$\rho_r^*\left(\left\|\nabla f(w_{m_k}) - \nabla f(u_{i,n_k})\right\|\right) \to 0 \quad \text{as } k \to \infty,$$

for each $i \in \{1, 2, \dots, N\}$. Thus, following the method of proof of Case 1, we obtain $d(w_{m_k}, T_i w_{m_k}) \to 0$ as $k \to \infty$, and hence we obtain

$$\limsup_{k\to\infty} \langle \nabla f(u) - \nabla f(p), w_{m_k} - p\rangle \leq 0. \tag{3.13}$$

Then, from (3.8), we get

$$D_f(p, x_{m_k+1}) \leq (1 - \alpha_{m_k}) D_f(p, x_{m_k}) + \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), w_{m_k} - p\rangle. \tag{3.14}$$

Now, since $D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1})$, inequality (3.14) implies that

$$\begin{aligned}
\alpha_{m_k} D_f(p, x_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) \\
&\quad + \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), w_{m_k} - p\rangle \\
&\leq \alpha_{m_k} \langle \nabla f(u) - \nabla f(p), w_{m_k} - p\rangle.
\end{aligned}$$

Thus, we get

$$D_f(p, x_{m_k}) \leq \langle \nabla f(u) - \nabla f(p), w_{m_k} - p\rangle. \tag{3.15}$$

Then, from (3.15) and (3.13), we obtain $D_f(p,x_{m_k}) \to 0$ as $k \to \infty$. This, together with (3.14), gives $D_f(p,x_{m_k+1}) \to 0$ as $k \to \infty$. But $D_f(p,x_k) \leq D_f(p,x_{m_k+1})$ for all $k \in \mathbb{N}$, and hence we obtain $x_k \to p \in \mathcal{F}$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$ and the proof is complete. □

If in Theorem 3.2, we assume that $N = 1$, then we get the following corollary.

**Corollary 3.3** *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $E$. Let $C$ be a nonempty, closed, and convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $T : C \to CB(C)$ be a Bregman relatively nonexpansive mapping such that $F(T)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} w_n = P^f_C \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)), \\ x_{n+1} = \nabla f^*(\beta \nabla f(w_n) + (1-\beta)\nabla f(u_n)), \quad u_n \in Tw_n, \forall n \geq 0, \end{cases} \tag{3.16}$$

*where $\{\alpha_n\} \subset (0,1)$ and $\beta \in (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$.*

If, in Theorem 3.2, we assume that each $T_i$, $i = 1,2,\ldots,N$ is a single-valued Bregman relatively nonexpansive mapping, we get the following corollary.

**Corollary 3.4** *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $E$. Let $C$ be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $T_i : C \to C$, for $i = 1,2,\ldots,N$, be a finite family of Bregman relatively nonexpansive mappings such that $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} w_n = P^f_C \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)), \\ x_{n+1} = \nabla f^*(\beta_0 \nabla f(w_n) + \sum_{i=1}^{N} \beta_i \nabla f(T_i w_n)), \quad \forall n \geq 0, \end{cases} \tag{3.17}$$

*where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_i\}_{i=0}^{N} \subset [c,d] \subset (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{i=0}^{N} \beta_i = 1$. Then $\{x_n\}$ converges strongly to $p = P^f_{\mathcal{F}}(u)$.*

If, in Theorem 3.2, we assume that each $T_i$, $i = 1,2,\ldots,N$, is a multi-valued quasi-Bregman relatively nonexpansive mapping, we get the following corollary.

**Corollary 3.5** *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of $E$. Let $C$ be a nonempty, closed, and convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $T_i : C \to CB(C)$, for $i = 1,2,\ldots,N$, be a finite family of quasi-Bregman nonexpansive mappings with $F(T_i) = \widehat{F}(T_i)$, for each $i \in \{1,2,\ldots,N\}$. Suppose that $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} w_n = P^f_C \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)), \\ x_{n+1} = \nabla f^*(\beta_0 \nabla f(w_n) + \sum_{i=1}^{N} \beta_i \nabla f(u_{i,n})), u_{i,n} \in T_i w_n, \quad \forall n \geq 0, \end{cases} \tag{3.18}$$

*where* $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_i\}_{i=0}^N \subset [c,d] \subset (0,1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\sum_{i=0}^N \beta_i = 1$. *Then* $\{x_n\}$ *converges strongly to* $p = D^f_{\mathcal{F}}(u)$.

**Remark 3.6** (i) Theorem 3.2 improves and extends the corresponding results of Homaeipour and Razani [33] and Zegeye and Shahzad [34] to the class of multi-valued Bregman relatively nonexpansive mappings in a reflexive real Banach spaces. (ii) The requirement that *the interior of F is nonempty* is dispensed with.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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